Journal of Statistical Physics, Vol. 99, Nos. 1/2, 2000

Asymmetric Particle Systems on $\mathbb{R}$

J. Krug[1] **and J. García**[2]

Received July 9, 1999; final December 1, 1999

We study interacting particle systems on the real line which generalize the Hammersley process [D. Aldous and P. Diaconis, *Prob. Theory Relat. Fields* **103**:199–213 (1995)]. Particles jump to the right to a randomly chosen point between their previous position and that of the forward neighbor at a rate which may depend on the distance to the neighbor. A class of models is identified for which the invariant particle distribution is Poisson. The bulk of the paper is devoted to a model where the jump rate is constant and the jump length is a random fraction r of the distance to the forward neighbor drawn from a probability density $\phi(r)$ on the unit interval. This is a special case of the random average process of Ferrari and Fontes [*El. J. Prob.* **3** (1998)]. The discrete-time version of the model has been considered previously in the context of force propagation in granular media [S. N. Coppersmith *et al.*, *Phys. Rev. E* **53**:4673 (1996)]. We show that the stationary two-point function of particle spacings factorizes for any choice of $\phi(r)$. Under the assumption that this implies pairwise independence, the invariant density of interparticle spacings for the case of uniform $\phi(r)$ is found to be a gamma distribution with parameter ν, where $\nu = 1/2$, 1, and 2 for continuous-time, backward sequential, and discrete-time dynamics respectively. A heuristic derivation of a nonlinear diffusion equation is presented, and the tracer diffusion coefficient is computed for arbitrary $\phi(r)$ and different types of dynamics.

KEY WORDS: Interacting particle systems; random average process; invariant product measures; discrete-time dynamics; hydrodynamic limit; single-file diffusion; granular packings.

1. INTRODUCTION AND OUTLINE

In this paper we are concerned with systems of interacting particles moving on the real line. The models of interest can be described as follows: Let

[1] Fachbereich Physik, Universität GH Essen, D-45117 Essen, Germany.
[2] Instituto de Matemática e Estatistica, Universidade de São Paulo, 05315-970 São Paulo, Brazil.

$x_i \in \mathbb{R}$ denote the position of the ith particle. In an elementary move particle i jumps to the right to a position $x_i + \delta_i$ between x_i and $x_{i+1} > x_i$. In the absence of a lattice spacing, there are two natural ways of setting the scale for the jump distance δ_i: It can be imposed externally through the choice of a fixed probability density $f_i(\delta_i)$, in which case moves with $\delta_i > x_{i+1} - x_i$ have to be rejected, or the scale can be set by the gap or "headway"

$$u_i = x_{i+1} - x_i \tag{1.1}$$

in front of particle i by letting f_i depend on the configuration $\mathcal{U} \equiv \{u_i\}_{i \in \mathbb{Z}}$ as

$$f_i(\delta_i \mid \mathcal{U}) = u_i^{-1} \phi(\delta_i / u_i) \tag{1.2}$$

where $\phi(r)$ is a probability density with support on the unit interval. Equation (1.2) implies that the jump length δ_i is a random fraction r of the headway u_i. The rate for the move is a function $\gamma(u_i)$ of the headway. The moves are executed in continuous time (in which case each particle is equipped with an exponential clock) or in discrete time; in the latter case the particle positions are updated either in parallel, or sequentially by going through the system against the direction of particle motion. A model is defined by specifying the functions $f_i(\delta)$ and $\gamma(u)$ as well as the type of dynamics (continuous time, parallel or sequential).

Two equivalent representations of the dynamics will prove to be useful. In terms of the headway variables u_i the particle configuration may be visualized as a system of sticks located at the sites i of the integer lattice, u_i being the length of stick i. In an elementary move a fraction δ_i of stick i is broken off and added to stick $i-1$.[30, 26] Alternatively, the particle positions $x_i(t)$ can be taken to define the height of a one-dimensional interface above the point i. The asymmetric particle motion translates into a growth process, and the fact that particles cannot pass each other implies that the interface is a monotonically increasing staircase ($x_{i+1} - x_i > 0$) at all times. We will refer to these two viewpoints as the stick representation and the interface representation, respectively.

For continuous time dynamics, a jump length distribution of the type (1.2) with ϕ uniform, and $\gamma(u) = u$ the model reduces to the Hammersley process discussed in ref. 1. In this case the invariant distribution of particle positions is Poisson. Here we are interested in obtaining similar results for other choices of f_i and γ, and other types of dynamics. Our motivation is mainly conceptual: While a wealth of results[22, 32, 15, 23, 32] are available for particle systems on the integer lattice such as the asymmetric simple exclusion process,[31] little is known analytically for the case of continuous particle positions, although motion on the real line appears naturally e.g., in applications to highway traffic.[16–18]

An important simplifying feature of the asymmetric exclusion process is the existence of stationary product measures. Here the analogous desirable property is the product form

$$\mathscr{P}(\mathscr{U}) = \prod_i P(u_i) \tag{1.3}$$

for the stationary probability of a configuration $\mathscr{U}$ of particle headways. Therefore a primary goal will be to find nontrivial examples of asymmetric particle systems on $\mathbb{R}$ for which (1.3) holds.

We provide an outline of the paper. In the next section we explore the conditions for a Poisson distribution of particle positions (corresponding to an exponential distribution of interparticle spacings in (1.3)) to be invariant for continuous time dynamics. Our strategy is to consider a finite number N of particles moving on a ring of length L, and to demand that the stationary measure gives the same weight to all allowed configuration; this then implies a Poisson measure for $N, L \to \infty$ at fixed density $\rho = N/L$. Provided the jump rate γ is independent of the headway, we find that the Poisson measure is invariant for *arbitrary* externally imposed (i.e., configuration and particle independent) jump length distributions $f(u)$. On the other hand, if the jump length is scaled to the headway as in (1.2), the Poisson measure is stationary only for a one-parameter family of power law functions ϕ and γ, which have been identified previously in the context of (symmetric) stick models.[10]

Sections 3 and 4, which constitute the main part of the paper, are devoted to models with constant jump rate, $\gamma \equiv 1$ independent of the headway, and jump length distributions of the type (1.2). In the interface representation these belong to the class of random average processes (RAP) studied by Ferrari and Fontes:[12] The particle position x_i' after the move is an average

$$x_i' = r x_i + (1-r)\, x_{i+1} \tag{1.4}$$

of the previous positions x_i, x_{i+1}, with a random weight $r \in [0, 1]$ drawn from the probability density $\phi(r)$. We therefore refer to these models as Asymmetric Random Average Processes (ARAP). Discrete time ARAP's have been introduced previously to model force fluctuations in random bead packs.[24, 7, 6] In that context the headway $u_i(t)$ represents the (scaled) force supported by bead i at depth t below the surface of a two-dimensional packing (see Section 3.2.1).

In Section 3.1.1 we show, for the case of continuous time dynamics, that the two-point correlation function of particle headways $\langle u_i u_j \rangle$ factorizes in the stationary state for any choice of $\phi(r)$, and obtain the expression

$$\langle u^2 \rangle - \langle u \rangle^2 = \frac{\mu_2}{\rho^2(\mu_1 - \mu_2)} \tag{1.5}$$

for the stationary variance of headways in terms of the moments

$$\mu_n = \int_0^1 dr\, r^n \phi(r) \tag{1.6}$$

of $\phi(r)$ and the particle density ρ. Similar results for the discrete time models are derived in Section 3.2.

More detailed information about the stationary headway distribution can be obtained when $\phi(r)$ is the uniform distribution on $[0, 1]$. Assuming that the factorization property of the two-point function implies pairwise independence of the u_i, we derive and solve stationarity conditions for their moments, which show that the invariant density of headways (normalized to $\langle u_i \rangle = 1$) takes the form of a gamma distribution,

$$P_\nu(u) = \frac{\nu^\nu}{\Gamma(\nu)} u^{\nu-1} e^{-\nu u} \tag{1.7}$$

where the parameter ν depends on the dynamics: For continuous time dynamics $\nu = 1/2$, while sequential and parallel dynamics yield $\nu = 1$ and 2, respectively. The result for parallel dynamics has been previously derived by Coppersmith *et al.*,[7] who also gave an explicit proof of the factorization property (1.3). Equation (1.7) implies *bunching* of particles (enhanced density fluctuations compared to the Poisson measure) for continuous time dynamics ($\nu = 1/2$) and *antibunching* for parallel dynamics ($\nu = 2$). The associated nontrivial particle–particle correlations are explicitly computed in Section 3.3.

Based on numerical simulations, we conjecture that the stationary single particle headway distribution is exactly given by (1.7) for all three types of dynamics. For continuous time dynamics and a finite number of particles on a ring the assumption of an invariant product measure is examined in Section 3.1.2. Surprisingly, we find that the product measure is *not* invariant for the ARAP, although it is invariant for a related symmetric stick model. This conclusion agrees with recent results for the infinite system obtained by Rajesh and Majumdar.[26]

Section 4 is devoted to the large scale, long time behavior of the ARAP. We derive a hydrodynamic equation of singular diffusion type, and compute the tracer diffusion coefficient using a Langevin approach. Since these results depend only on the stationary two-point function of headways, they are valid for any choice of the jump length distribution $\phi(r)$. Finally, some conclusions and open questions are formulated in Section 5.

2. MODELS WITH INVARIANT POISSON MEASURES

2.1. Constant Invariant Measure on the Ring

In this section we want to identify continuous time dynamics which leave a Poisson distribution of particle positions invariant. For this purpose we first consider N particles moving in continuous time on a ring of length L, with density $\rho = N/L$. Allowed headway configurations then satisfy the constraint

$$\sum_{i=1}^{N} u_i = L \tag{2.1}$$

and the product measure (1.3) is required to hold on the set of configurations defined by (2.1). For an exponential distribution $P(u) \sim e^{-\rho u}$ this implies that all allowed headway configurations carry the same weight $\Omega(N, L)^{-1}$, where

$$\Omega(N, L) = \frac{L^{N-1}}{(N-1)!} \tag{2.2}$$

denotes the volume of the set, i.e., the invariant measure is *constant* on allowed configurations. It is straightforward to check that this implies Poisson measure in the limit $N, L \to \infty$ at fixed density ρ. For example, the distribution of a single headway on the ring is given by

$$P_{N,L}(u) = \frac{\Omega(N-1, L-u)}{\Omega(N, L)} \to \rho e^{-\rho u}, \qquad N, L \to \infty \tag{2.3}$$

while the joint distribution of the headways of two neighboring particles is

$$P_{N,L}(u_i, u_{i+1}) = \frac{\Omega(N-2, L-u_i-u_{i+1})}{\Omega(N, L)} \to \rho^2 e^{-\rho(u_i+u_{i+1})}, \qquad N, L \to \infty \tag{2.4}$$

A similar argument can be carried out for the probability distribution of the particle positions on the ring.

Invariance of the constant measure requires the total transition rates for going into and out of any configuration to balance. This yields the condition

$$\sum_{i=1}^{N} \int_0^{u_{i-1}} dw\, f_i(w \mid \mathscr{U}^{(i)}(w))\, \gamma(u_i + w) = \sum_{i=1}^{N} \int_0^{u_i} dw\, f_i(w \mid \mathscr{U})\, \gamma(u_i) \qquad (2.5)$$

for any configuration $\mathscr{U}$, with the configuration $\mathscr{U}^{(i)}(w) = \{u_j^{(i)}(w)\}_{j \in \mathbb{Z}}$ defined through

$$u_j^{(i)}(w) = \begin{cases} u_i + w: & j = i \\ u_{i-1} - w: & j = i-1 \\ u_j: & \text{else} \end{cases} \qquad (2.6)$$

and periodic boundary conditions implied in the summation over i. Note the upper integration limits, which ensure that particles cannot pass each other ($\delta_i \leqslant u_i$). Two examples of dynamics which satisfy (2.5) will be given in the following.

2.2. Configuration-Independent Jump Length Distributions

If the jump rate γ is independent of headway, the invariance condition (2.5) is seen to hold for *any* jump length distribution $f(w)$ which is independent of the configuration and of the particle label i. The stationary speed $\bar{v}$ of particles at density ρ is then computed from

$$\bar{v} = \gamma \rho \int_0^{\infty} du\, e^{-\rho u} \int_0^u dw\, w f(w) \qquad (2.7)$$

and the current follows from $j(\rho) = \rho \bar{v}(\rho)$. For example, for jump lengths chosen uniformly in the unit interval one finds

$$j(\rho) = \frac{\gamma}{\rho} [1 - (1 + \rho)\, e^{-\rho}] \qquad (2.8)$$

It should be noted that in general the Poisson distribution is not the unique invariant measure. For example, if $f(w) = 0$ for w less than some minimum jump length a, then all configurations with $u_i < a$ for all i are trivially invariant. Numerical simulations indicate, however, that such "absorbing" states are typically not reached, even if the system is started very close to them. If $f(w) = \delta(w - 1)$ *and* the particles are started on the integer lattice, the model reduces to the asymmetric exclusion process, which has a geometric (rather than exponential) headway distribution.

2.3. Scale-Invariant Models

When the scale of the jumps is set by the headways, inserting (1.2) into (2.5) and requiring the terms on both sides to cancel pairwise yields the following integral equation connecting the functions ϕ and γ,

$$\int_0^u dw\, \gamma(u'+w)\, \frac{\phi(w/(u'+w))}{u'+w} = \gamma(u) \tag{2.9}$$

which should be true for all u, u'. Taking the derivative with respect to u this becomes a differential equation for γ,

$$\frac{d\gamma}{du} = \frac{\gamma(v)\, \phi(u/v)}{v} \tag{2.10}$$

with $v = u + u' \geqslant u$. Setting in particular $v = u$ we see that γ has to be a power law function,

$$\gamma(u) = \gamma_0 u^{\alpha - 1} \tag{2.11}$$

where $\gamma_0 > 0$ is a constant and $\alpha = 1 + \phi(1)$. Using (2.10) the jump length distribution is then found to be also a power law,

$$\phi(v) = (\alpha - 1)\, v^{\alpha - 2} \tag{2.12}$$

Normalizability of ϕ requires $\alpha > 1$.

Equations (2.11) and (2.12) define a one-parameter family of models for which the Poisson distribution of positions is invariant for an arbitrary number of particles N, the Hammersley process being given by $\alpha = 2$. The corresponding *symmetric* stick models, in which the broken-off piece is distributed with equal probability to the left or right neighbor, were considered by Feng *et al.*[10] Since γ is a power law, these models are *scale invariant* in the sense that the average particle spacing $\langle u_i \rangle = 1/\rho$ is the only length scale in the problem. Therefore also the stationary particle current j is a power law function of the density. To compute it, we note that the average particle speed is given by

$$\bar{v} = \langle \gamma(u_i)\, \delta_i \rangle = \rho \int_0^\infty du\, e^{-\rho u} \gamma(u)\, u \int_0^1 dv\, v\phi(v) = \gamma_0 (1 - 1/\alpha)\, \Gamma(\alpha + 1)\, \rho^{-\alpha} \tag{2.13}$$

and therefore

$$j(\rho) = \rho \bar{v} = \gamma_0 (1 - 1/\alpha)\, \Gamma(\alpha + 1)\, \rho^{1 - \alpha} \tag{2.14}$$

3. ASYMMETRIC RANDOM AVERAGE PROCESSES

The asymmetric random average process is a scale-invariant model characterized by a jump length distribution of type (1.2), and a constant jump rate $\gamma \equiv \gamma_0 = 1$. The discussion is phrased most naturally in the stick representation, and begins with the continuous time models.

3.1. Continuous-Time Dynamics

3.1.1. Stationary Headway Correlations. Consider first the time evolution of the second moment $\langle u_i^2 \rangle$. In a small time interval Δt two processes affecting u_i may occur: A random fraction δ_i of u_i may be lost to $i-1$, and a random fraction δ_{i+1} of u_{i+1} may be gained from $i+1$. Both processes occur with probability Δt. Thus

$$\langle u_i^2 \rangle (t+\Delta t) = \Delta t [\langle (u_i - \delta_i)^2 \rangle + \langle (u_i + \delta_{i+1})^2 \rangle] + (1 - 2\Delta t) \langle u_i^2 \rangle (t) \tag{3.1}$$

Stationarity then implies

$$-2\langle \delta_i u_i \rangle + \langle \delta_i^2 \rangle + 2\langle \delta_{i+1} u_i \rangle + \langle \delta_{i+1}^2 \rangle = 0 \tag{3.2}$$

Since $\delta_j = r_j u_j$ where r_j is an independent random variable with mean μ_1 and second moment μ_2, we have that $\langle \delta_i u_i \rangle = \mu_1 \langle u_i^2 \rangle$, $\langle \delta_i^2 \rangle = \langle \delta_{i+1}^2 \rangle = \mu_2 \langle u_i^2 \rangle$ and $\langle \delta_{i+1} u_i \rangle = \mu_1 \langle u_i u_{i+1} \rangle$. Thus (3.2) becomes

$$(\mu_1 - \mu_2) \langle u_i^2 \rangle = \mu_1 \langle u_i u_{i+1} \rangle \tag{3.3}$$

Similarly for the general two-point function $C_k \equiv \langle u_i u_{i+k} \rangle$ we obtain the stationarity condition

$$\mu_1 (C_{k+1} + C_{k-1} - 2C_k) = \mu_2 C_0 (\delta_{k,1} + \delta_{k,-1} - 2\delta_{k,0}) \tag{3.4}$$

where translational invariance and symmetry ($C_k = C_{-k}$) of the correlations has been used. Solving Eq. (3.4) starting from $k=0$ one finds

$$C_k = [1 - (\mu_2/\mu_1)(1 - \delta_{k,0})] \; C_0 \tag{3.5}$$

Imposing the boundary condition $\lim_{k \to \infty} C_k = \langle u_i \rangle^2 = 1/\rho^2$ for an infinite system of density ρ, Eq. (3.5) then shows that the two-point function factorizes for any $k \geqslant 1$ and the variance of headways is given by (1.5).

3.1.2. Stationary Headway Distribution for Uniform $\phi(r)$. We now specialize to the case when the distribution of scaled jump lengths $\phi(r)$ is uniform in $[0, 1]$, and assume that the factorization property which was verified above for the two-point function implies the pairwise independence of the u_i. Then the stationarity condition for the nth moment

$$\langle (u_i+\delta_{i+1})^n \rangle + \langle (u_i-\delta_i)^n \rangle = 2\langle u_i^n \rangle \tag{3.6}$$

yields (the index i of u_i is now dropped)

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} [\langle u^{n-k} \rangle \langle u^k \rangle + (-1)^k \langle u^n \rangle] = 2\langle u^n \rangle \tag{3.7}$$

which can be rewritten as a recursion relation,

$$\langle u^n \rangle = \frac{n+1}{n-1} \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{k+1} \langle u^{n-k} \rangle \langle u^k \rangle \tag{3.8}$$

Evaluating this expression for $n = 1,..., 5$ we find that the relation

$$\langle u^n \rangle = \left[\prod_{k=1}^{n} (2k-1) \right] \langle u \rangle^n \tag{3.9}$$

appears to hold, which is characteristic of the gamma distribution (1.7) with parameter $v = 1/2$.

To prove it, we first insert (3.9) into (3.8), and obtain

$$\binom{2n}{n} = \frac{n+1}{n-1} \sum_{k=1}^{n-1} \frac{1}{k+1} \binom{2k}{k} \binom{2(n-k)}{n-k} \tag{3.10}$$

This can be verified using the binomial expansion

$$\tfrac{1}{2}(1-4x)^{-1/2} = \tfrac{1}{2} + \sum_{k=1}^{\infty} \binom{2k-1}{k-1} x^k \tag{3.11}$$

Integrating with respect to x we also have

$$-\frac{1}{4}(1-4x)^{1/2} = -\frac{1}{4} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{1}{k+1} \binom{2k-1}{k-1} x^{k+1} \tag{3.12}$$

Since the product of the left hand sides is a constant, all coefficients of x^m with $m > 0$ in the series obtained by multiplying (3.11) and (3.12) must vanish. After rearranging terms this is seen to imply (3.10).

Table I. Numerical Estimates of the First Few Moments of the Stationary Headway Distribution for the ARAP with Uniform $\phi(r)$ and Different Kinds of Update[a]

Dynamics	$\langle u^2 \rangle$	$\langle u^3 \rangle$	$\langle u^4 \rangle$	$\langle u^5 \rangle$
Continuous	2.998 ± 1	15.02 ± 1	105.3 ± 2	947 ± 3
	(3)	(15)	(105)	(945)
Sequential	1.9997 ± 2	5.995 ± 2	23.93 ± 2	119.2 ± 2
	(2)	(6)	(24)	(120)
Parallel	1.4998 ± 1	2.996 ± 1	7.466 ± 3	22.26 ± 2
	(3/2)	(3)	(15/2)	(45/2)

[a] The data were obtained from simulations of systems of 2×10^5 particles which were started from an ordered initial condition, $u_i = 1$ for all i, and allowed to evolve for 10^4 time steps. To extrapolate to $t \to \infty$, each run was fitted to Eq. (3.13), and the errors were estimated by taking an average over 10 runs (errors refer to the last digit shown). The numbers in parentheses are the conjectured values of the moments; for the case of parallel update these are known to be exact.[7] The remaining discrepancies are in fact largest for parallel update, and can probably be attributed to residual finite time effects.

In fact the relation (3.9) was first guessed on the basis of numerical simulations. Rather accurate numerical estimates for the stationary moments of u_i can be obtained by starting from an ordered initial condition ($u_i = 1$ for all i) and fitting the finite time data to the form

$$\langle u^n \rangle (t) = A_n + B_n t^{-1/2} \tag{3.13}$$

which is suggested by the fluctuation theory of Section 4.2 (see Eq. (4.21)). The results shown in Table I strongly indicate that the stationary single particle headway distribution is exactly given by the $\nu = 1/2$ gamma distribution.

To test for the existence of an invariant product measure, we proceed as in Section 2 and consider a finite number N of particles on a ring. The condition for the product measure (1.3), restricted to the set (2.1) of allowed configurations, to be invariant now reads

$$\sum_{i=1}^{N} \int_0^{u_{i-1}} \frac{dw}{u_i + w} \frac{P(u_i + w)\, P(u_{i-1} - w)}{P(u_i)\, P(u_{i-1})} = \sum_{i=1}^{N} \gamma(u_i) = N \tag{3.14}$$

Inserting the gamma distribution with parameter $\nu = 1/2$ (Eq. (1.7)) and noting that

$$\int_0^{v} dw (u + w)^{-3/2} (v - w)^{-1/2} = \frac{2\sqrt{v/u}}{u + v} \tag{3.15}$$

the condition (3.14) becomes

$$\sum_{i=1}^{N} \frac{2u_{i-1}}{u_i + u_{i-1}} = N \tag{3.16}$$

with periodic boundary conditions, $u_{-1} = u_N$. Equation (3.16) is satisfied for $N = 2$, but not for general N. We conclude that the product measure (1.3) is *not* invariant for N different from 2. It is however the exact invariant measure for the *symmetric* stick process obtained by transferring the piece broken off stick i to $i-1$ or $i+1$ with equal probability. Indeed, in that case the left hand side of (3.16) becomes

$$\sum_{i=1}^{N} \frac{u_{i-1}}{u_i + u_{i-1}} + \frac{u_{i+1}}{u_i + u_{i+1}} = N \tag{3.17}$$

While these arguments are restricted to finite systems, the conclusions agree with calculations carried out for the infinite system by Rajesh and Majumdar.[26] Specifically, they show that the product measure ansatz for the continuous time ARAP breaks down at the level of three-point correlations, but is exact for the symmetric stick model.

3.2. Discrete-Time Dynamics

3.2.1. Parallel Update. A discrete time version of the ARAP is obtained by writing

$$u_i(t+1) = u_i(t) - \delta_i(t) + \delta_{i+1}(t) \tag{3.18}$$

where $\delta_j = r_j u_j$ with independent random numbers r_j distributed according to the density $\phi(r)$. This is closely related to a model introduced by Coppersmith, Liu, Majumdar, Narayan and Witten for the description of force fluctuations in bead packs.[7] To see the connection, let $W(i, t)$ denote the weight supported by bead i in the tth layer below the (free) surface of the packing. The key assumption of the model is that the beads are arranged on a regular lattice, and that each bead transfers its weight to exactly M beads in the layer below. The fraction $q_{ij}(t) \in [0, 1]$ of the weight of bead i in layer t which is transferred to bead j in layer $t+1$ defines a matrix with random entries subject to the constraint $\sum_j q_{ij}(t) = 1$. Assigning unit mass to each bead, the weights evolve according to

$$W(j, t+1) = 1 + \sum_i q_{ij}(t)\, W(i, t) \tag{3.19}$$

For large t all weights increase linearly with t, which suggests to introduce normalized variables $U(i, t) = W(i, t)/t$. Specializing to a two-dimensional lattice where the beads are labeled such that bead i is connected to beads i and $i+1$ in the layer below, we see that for $t \to \infty$ the evolution of the $U(i, t)$ reduces to (3.18) with the identification $q_{ii} = 1 - r_i$ and $q_{i+1i} = r_{i+1}$. In the context of beak packs q_{ii} and q_{i+1i} should have the same distribution, and hence strict equivalence between the two models holds only when $\phi(r)$ is symmetric around $r = 1/2$.

Let us first show that the stationary two-point headway correlations factorize for any $\phi(r)$. Proceeding as above in Section 3.1.1, we obtain the stationarity condition

$$(\mu_1 - \mu_1^2)(C_{k+1} + C_{k-1} - 2C_k) = (\mu_2 - \mu_1^2)\ C_0(\delta_{k,1} + \delta_{k,-1} - 2\delta_{k,0}) \tag{3.20}$$

with the solution

$$C_k = [1 - (\mu_2 - \mu_1^2)/(\mu_1 - \mu_1^2)(1 - \delta_{k,0})]\ C_0 \tag{3.21}$$

As in the continuous time case this implies factorization for $k \geqslant 1$ in the infinite system, with the stationary variance of headways given by

$$\langle u^2 \rangle - \langle u \rangle^2 = \frac{\mu_2 - \mu_1^2}{\rho^2(\mu_1 - \mu_2)} \tag{3.22}$$

For the case of a uniform distribution $\phi(r)$, Coppersmith *et al.*[7] (see also refs. 25 and 26) have shown explicitly that the stationary measure takes the product form (1.3), with the headway distribution $P(u)$ given by the gamma distribution (1.7) with $v = 2$. The latter is easily derived along the lines of Section 3.1.2. Under the assumption of pairwise independence, the stationarity condition for general moments $\langle u_i^n \rangle$ now reads

$$\langle u^n \rangle = \frac{1}{(n-1)(n+2)} \sum_{k=1}^{n-1} \binom{n+2}{k+1} \langle u^{n-k} \rangle \langle u^k \rangle \tag{3.23}$$

A straightforward computation shows that this is solved by the expression

$$\langle u^n \rangle = 2^{-n}(n+1)!\ \langle u \rangle^n \tag{3.24}$$

for the moments of the gamma distribution (1.7) with parameter $v = 2$.

3.2.2. Ordered Sequential Update. In the context of traffic modeling[9, 27] it has been found useful to implement a different kind of discrete time dynamics, in which the particles are moved one by one, in the order of their positions in the system. This *ordered sequential update* can proceed either in the direction of particle motion (forward update) or against it (backward update). For the ARAP it is easy to see that the forward update is equivalent to the parallel dynamics discussed in Section 3.2.1, however the backward update is not.

In the stick representation, backward sequential update implies that stick i first receives a random fraction of stick $i+1$, placing it in an intermediate state of length u_i', and subsequently transfers a random fraction δ_i' of u_i' to stick $i-1$. It is important to note that, at the time of transfer of mass to stick i, stick $i+1$ has already received mass from $i+2$ and thus the amount transferred from $i+1$ to i is a random fraction of $u_{i+1}' > u_{i+1}$. The dynamics therefore proceeds in two steps,

$$u_i'(t) = u_i(t) + \delta_{i+1}'(t) \tag{3.25}$$

$$u_i(t+1) = u_i'(t) - \delta_i'(t) \tag{3.26}$$

where δ_j' is a random fraction of u_j'. Taking the average of both sides of (3.25) or (3.26) the stationary mean of u_i' is seen to be

$$\langle u' \rangle = \frac{\langle u \rangle}{1-\mu_1} = \frac{1}{\rho(1-\mu_1)} \tag{3.27}$$

Equation (3.26) implies the relation

$$C_k = [(1-\mu_1)^2 + (\mu_2 - \mu_1^2)\,\delta_{k,0}]\; C_k' \tag{3.28}$$

between the stationary two-point functions C_k of u_i and C_k' of u_i'. Using (3.25) it is easy to show that the stationarity condition for C_k' is identical to the condition (3.20) obtained in the case of parallel update. Therefore also C_k' factorizes in the infinite system, and through (3.28) this property carries over to C_k. For the stationary variance of the backward sequential update model we find the expression

$$\langle u^2 \rangle - \langle u \rangle^2 = \frac{\mu_2 - \mu_1^2}{\rho^2(1-\mu_1)(\mu_1-\mu_2)} \tag{3.29}$$

Turning to the stationary headway probability distribution $P(u)$, we again assume pairwise independence and note the functional equation

$$P(u) = \int_0^1 dr\; r^{-1} \phi(1-r)\; P'(u/r) \tag{3.30}$$

relating $P(u)$ to the distribution $P'(u')$ of the intermediate state headway. For uniform $\phi(r)$ the stationarity condition for the nth moment of u_i' then reads

$$\langle (u_i')^n \rangle = \langle (u_i + \delta_{i+1}')^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \langle (u_i')^k \rangle \langle u_i^{n-k} \rangle \tag{3.31}$$

Using the relation $\langle (u')^n \rangle = (n+1) \langle u^n \rangle$ obtained from (3.30) this reduces to

$$\langle u^n \rangle = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \langle u^k \rangle \langle u^{n-k} \rangle \tag{3.32}$$

which is solved by setting $\langle u^n \rangle = n! \langle u \rangle^n$. We conclude that $P(u)$ is an exponential distribution (a gamma distribution (1.7) with $v = 1$). This is confirmed by the numerical data shown in Table I.

From (3.30) the distribution of the intermediate state headway is found to be a $v = 2$ gamma distribution with mean $2\langle u \rangle = 2/\rho$,

$$P'(u) = \rho^2 u e^{-\rho u} \tag{3.33}$$

Given the equivalence between the intermediate state headway and the headway for parallel update which we found on the level of the two-point function, it is no surprise that (3.33) is identical, up to a scale factor, to the headway distribution $P(u)$ for parallel dynamics.

3.3. Particle–Particle Correlations

In this section we illustrate how the product measure (1.3) with the headway distribution (1.7) translates into nontrivial particle–particle correlations when $v \neq 1$. For example, the probability density $g(x)$ for finding a particle at x, conditioned on having a particle at the origin, can be written as

$$g(x) = \sum_{n=1}^{\infty} P_n(x) \tag{3.34}$$

where $P_n(x)$ is the probability density for the nth particle to be at x when the 0th is at the origin or, equivalently, the probability that $\sum_{i=0}^{n-1} u_i = x$. The P_n are obtained iteratively from $P_1(x) = P(x)$ through the convolution

$$P_n(x) = \int_0^x dy \, P_{n-1}(y) \, P(x-y) \tag{3.35}$$

Inserting the gamma distributions (1.7) with parameters $\nu = 1/2$ and $\nu = 2$, one finds that

$$P_n(x) = \rho(\Gamma(n/2)\, 2^{n/2})^{-1}\, (\rho x)^{n/2 - 1}\, e^{-\rho x/2} \tag{3.36}$$

for the continuous time case, and

$$P_n(x) = \frac{2^{2n}\rho}{(2n-1)!}\, (\rho x)^{2n-1}\, e^{-2\rho x} \tag{3.37}$$

for parallel dynamics.

In the parallel case the evaluation of the sum (3.34) is straightforward, and yields the expression

$$g(x) = \rho(1 - e^{-4\rho x}) \tag{3.38}$$

for the correlation function, which explicitly displays the tendency of particles to avoid each other at distances short compared to $1/\rho$.

To compute (3.34) with the P_n given by (3.36), it is useful to write g as the sum of two contributions g_{even} and g_{odd} from even and odd n, respectively. One finds that $g_{\text{even}}(x) = \rho/2$ independent of x, while the odd part can be brought into the form

$$g_{\text{odd}}(x) = P_1(x) + \frac{\rho}{2\sqrt{\pi}}\, e^{-\rho x/2} \sum_{m=1}^{\infty} \frac{(m-1)!}{(2m-1)!}\, (\sqrt{2\rho x})^{2m-1} \tag{3.39}$$

To sum the series we write $(m-1)! = \int_0^\infty dz\, z^{m-1} e^{-z}$ and interchange the summation over m with the integration over z. This yields finally

$$g(x) = \sqrt{\frac{\rho}{2\pi x}}\, e^{-\rho x/2} + \frac{\rho}{2}\,(1 + \text{erf}\,\sqrt{\rho x/2}) \tag{3.40}$$

with the error function $\text{erf}(z) = (2/\sqrt{\pi}) \int_0^z dt\, e^{-t^2}$. For $x \to 0$ the correlation function is dominated by $P_1(x)$ and correspondingly diverges as $1/\sqrt{x}$, reflecting the tendency of particles to bunch together in the continuous time case. For $x \to \infty$ $g(x)$ decays somewhat faster than exponentially, as

$$g(x) - \rho \approx \frac{\rho}{\sqrt{2\pi}\,(\rho x)^3}\, e^{-\rho x/2} \tag{3.41}$$

Alternatively the correlations between particles can be characterized through the variance $(\Delta N_L)^2$ of the number of particles N_L in an interval of size L. When L is small compared to the mean interparticle spacing N_L is either 0 or 1, and $(\Delta N_L)^2 = \rho L$. For $L \gg 1/\rho$ a central limit argument shows that

$$(\Delta N_L)^2 \approx \chi L \tag{3.42}$$

where the "compressibility" χ (defined in analogy with equilibrium systems[32]) is given by

$$\chi(\rho) = \rho^3(\langle u^2 \rangle - \langle u \rangle^2) = \rho/v \tag{3.43}$$

with the parameter v of the headway distribution (1.7). Thus the slope of $(\Delta N_L)^2$ versus L changes from ρ for $L \ll 1/\rho$ to ρ/v for $L \gg 1/\rho$, reflecting the increase (decrease) of particle number fluctuations for continuous time (parallel) dynamics, respectively. The compressibility is related to the pair correlation function (3.34) through

$$\chi = \rho \left(1 + \int_{-\infty}^{\infty} dx(g(x) - \rho)\right) \tag{3.44}$$

4. LARGE-SCALE DYNAMICS OF THE ARAP

4.1. Hydrodynamic Equation

The average particle speed $\bar{v}$ in the ARAP is inversely proportional to the density, hence the current $j = \rho\bar{v}$ is independent of ρ. The dynamics on the Euler scale $x \sim t$ is therefore trivial, and one expects a hydrodynamic equation of diffusion type.[32] A simple derivation will be given below. Throughout this section we consider a general scaled jump length distribution $\phi(r)$.

4.1.1. Continuous-Time Dynamics. In the continuous time case the ensemble averaged particle positions $X_i \equiv \langle x_i \rangle$ evolve according to the *linear* equations

$$\frac{dX_i}{dt} = \mu_1(X_{i+1} - X_i) \tag{4.1}$$

This problem has been studied previously in the context of crystal growth,[19] and the procedure can be directly applied to the present context.

To extract the long wavelength behavior, we introduce a scaling parameter[32, 15] ε and a smooth function $\xi(y, \tau)$ such that

$$X_i(t) = \xi(\varepsilon i, \varepsilon t) \tag{4.2}$$

Inserting this into (4.1) and expanding to second order in ε we obtain

$$\mu_1^{-1} \frac{\partial \xi}{\partial \tau} = \frac{\partial \xi}{\partial y} + \frac{\varepsilon}{2} \frac{\partial^2 \xi}{\partial y^2} \tag{4.3}$$

In the scaling limit $\varepsilon \to 0$ this becomes a first order equation which describes simple translation to the left.[12]

Here we will however postpone to take the limit, and first carry out a Lagrange transformation,[28, 19] which relates the Lagrangian description in terms of the particle positions $X_i(t)$ to the Eulerian evolution of the density field. The local density ρ near the position of particle i is estimated as $(X_{i+1} - X_i)^{-1}$, so using (4.2) we have the relation

$$\rho(\xi(y, \tau), \tau) = \varepsilon^{-1}(\partial \xi / \partial y)^{-1} \tag{4.4}$$

Differentiating this equation with respect to τ and using the evolution equation (4.3) for $\xi(y, \tau)$ one obtains, after some algebra,

$$\frac{\partial \rho}{\partial t} = \varepsilon \frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial x} \left(\frac{\mu_1}{2\rho^2} \right) \frac{\partial \rho}{\partial x} \tag{4.5}$$

The scaling factor ε cancels, and the *collective* diffusion coefficient is identified to be

$$D_{\mathrm{c}}(\rho) = \frac{\mu_1}{2\rho^2} \tag{4.6}$$

The ρ^{-2}-dependence is dictated by scale invariance: The typical jump length in a region of density ρ is $\bar{\delta} = \mu_1/\rho$, and $D_{\mathrm{c}} \sim \gamma \bar{\delta}^2 \sim \rho^{-2}$.

4.1.2. Discrete-Time Dynamics. For discrete parallel update Eq. (4.1) is replaced by

$$X_i(t+1) - X_i(t) = \mu_1 [X_{i+1}(t) - X_i(t)] \tag{4.7}$$

In the scaling limit $\varepsilon \to 0$ this results in the same coarse grained evolution equation (4.3), and thus also the nonlinear diffusion equation (4.5) is the same as in the continuous time case.

In the case of ordered sequential update one has to take into account that the new position of particle i is a random average of its old position and the *new* position of particle $i+1$, hence

$$X_i(t+1)-X_i(t)=\mu_1[X_{i+1}(t+1)-X_i(t)] \tag{4.8}$$

Making the ansatz $X_i(t)=i/\rho+\bar{v}t$, we see that the average particle speed is

$$\bar{v}=\frac{\mu_1}{\rho(1-\mu_1)}>\frac{\mu_1}{\rho} \tag{4.9}$$

The speedup compared to continuous time and parallel dynamics is due to the decrease of the local density near the update site, see ref. 27 for a discussion of similar effects in the asymmetric exclusion process. For the derivation of the hydrodynamic equation it is useful to incorporate the expected diffusive scaling from the outset and replace (4.2) by

$$X_i(t)=\xi(\varepsilon i, \varepsilon^2 t) \tag{4.10}$$

The expansion of (4.8) to second order in ε then yields

$$\left(\frac{1-\mu_1}{\mu_1}\right)\frac{\partial\xi}{\partial\tau}=\varepsilon^{-1}\frac{\partial\xi}{\partial y}+\frac{1}{2}\frac{\partial^2\xi}{\partial y^2} \tag{4.11}$$

As before, the drift term disappears under the Lagrange transformation based on the relation (4.4), and one obtains

$$\frac{\partial\rho}{\partial t}=\varepsilon^2\frac{\partial\rho}{\partial\tau}=\frac{\partial}{\partial x}\left(\frac{\mu_1}{2(1-\mu_1)\,\rho^2}\right)\frac{\partial\rho}{\partial x} \tag{4.12}$$

As far as the hydrodynamics is concerned, the different types of dynamics are seen to be equivalent up to a rescaling of time.

4.2. Tracer Diffusion

Hydrodynamic equations of diffusion type are usually associated with symmetric (unbiased) particle systems.[32] In one dimension the tracer diffusion coefficient in such systems typically vanishes, and the mean square displacement of a tagged particle grows subdiffusively as $t^{1/2}$.[2, 3] By contrast, the biased random average process shows normal tracer diffusion when started from a random initial condition and subdiffusive behavior when the initial configuration is ordered.[12] Here we provide a compact derivation of the two cases and compute the coefficient of the asymptotic law for different types of dynamics.

4.2.1. Langevin Approach for Continuous-Time Dynamics. We start the system in an initial condition without long wavelength fluctuations, such as $x_i(0)=i/\rho$, $i\in\mathbb{Z}$, and denote the positional fluctuation of particle i by

$$\zeta_i(t)=x_i(t)-\langle x_i\rangle=x_i(t)-x_i(0)-\bar{v}t \tag{4.13}$$

For the purpose of extracting the long time behavior of fluctuations, a Langevin approximation[13] to the dynamics of ζ_i is sufficient. Thus we add a phenomenological noise term $\eta_i(t)$ to the linear equation (4.1),

$$\frac{d\zeta_i}{dt}=\mu_1(\zeta_{i+1}-\zeta_i)+\eta_i \tag{4.14}$$

The noise is taken Gaussian with zero mean and covariance

$$\langle\eta_i(t)\,\eta_j(t')\rangle=\sigma\delta_{ij}\delta(t-t') \tag{4.15}$$

The noise strength σ will eventually be matched to the variance of particle headways.

Equation (4.14) is solved by introducing the Fourier transformed fluctuations

$$\hat{\zeta}(q,t)=\sum_{n\in\mathbb{Z}} e^{iqn}\zeta_n(t) \tag{4.16}$$

with wave numbers q in the first Brillouin zone $[-\pi,\pi]$, and the corresponding Fourier transformed noise

$$\hat{\eta}(q,t)=\sum_{n\in\mathbb{Z}} e^{iqn}\eta_n(t) \tag{4.17}$$

with covariance

$$\langle\hat{\eta}(q,t)\,\hat{\eta}(q',t')\rangle=2\pi\sigma\delta(q+q')\,\delta(t-t') \tag{4.18}$$

The most general quantity of interest is the variance of the displacement between particle i at time t and particle j at time t'. By translational invariance this depends only on $n=i-j$ and is given by the correlation function

$$G_n(t,t')=\langle(\zeta_0(t)-\zeta_n(t'))^2\rangle \tag{4.19}$$

Inserting (4.16) into (4.14), solving the equation for $\hat{\zeta}(q, t)$ and averaging over the noise according to (4.18) one arrives at the expression

$$G_n(t, t') = \frac{\sigma}{2\pi} \int_0^{\pi} \frac{dq}{\omega(q)} (2 - e^{-2\omega t} - e^{-2\omega t'} - 2\cos[qn - \mu(q)\, T](e^{-\omega |T|} - e^{-\omega T'})) \tag{4.20}$$

with $\omega(q) = \mu_1(1 - \cos(q))$, $\mu(q) = \mu_1 \sin(q)$, $T = t' - t$ and $T' = t' + t$.

The evaluation is straightforward in the relevant limiting cases. Consider first the variance of the headways at time $t = t'$. For large t (4.20) yields

$$G_1(t, t) \approx \frac{\sigma}{\mu_1} \left(1 - \frac{1}{2\sqrt{\pi \mu_1 t}}\right) \tag{4.21}$$

This allows us to identify the noise strength σ as

$$\sigma = \mu_1(\langle u^2 \rangle - \langle u \rangle^2) \tag{4.22}$$

and explicitly demonstrates the $1/\sqrt{t}$-approach to the stationary headway distribution alluded to in (3.13).

Next we focus on the dynamics of a single particle and set $n = 0$ in (4.20). If we fix the time increment $T = t' - t$ and let both t and $t' \to \infty$, G_0 represents the mean square displacement of a particle in the stationary regime. Evaluation of (4.20) gives $G_0(t, t') \approx \sigma\, |T|$, which shows that σ is precisely the tracer diffusion coefficient D_{tr}. Combining this with (4.22) and (1.5) we obtain

$$D_{\text{tr}} = \mu_1(\langle u^2 \rangle - \langle u \rangle^2) = \frac{\mu_1 \mu_2}{\rho^2(\mu_1 - \mu_2)} \tag{4.23}$$

In fact the first relation in (4.23) is easy to understand. The linear equation (4.3) shows that fluctuations in the particle positions drift backwards in "label space" $y = \varepsilon i$. This translates the stationary distance fluctuations into temporal fluctuations, with a conversion factor given by the drift speed μ_1. As was mentioned already, the existence of a nonvanishing tracer diffusion coefficient for models with a hydrodynamic equation of diffusion type is unusual in one dimension, since generically such an equation implies symmetric particle jumps, in which case the tracer particle displacement grows only subdiffusively due to the single file constraint.[2, 3] Here D_{tr} is nonzero because the particles move, at speed $\bar{v}$, relative to the (stationary) density fluctuations. A rigorous derivation of (4.23) has recently been presented by Schütz.[29]

Since the hydrodynamic equations in the two cases are identical, the argument leading to the first relation in (4.23) carries over directly to discrete parallel update, and using (3.22) we conclude that the tracer diffusion coefficient in this case is given by

$$D_{\mathrm{tr}}^{\mathrm{par}} = \frac{\mu_1(\mu_2 - \mu_1^2)}{\rho^2(\mu_1 - \mu_2)} \tag{4.24}$$

Similarly the expression

$$D_{\mathrm{tr}}^{\mathrm{seq}} = \frac{\mu_1(\mu_2 - \mu_1^2)}{\rho^2(1-\mu_1)^2\,(\mu_1 - \mu_2)} \tag{4.25}$$

is obtained for the backward sequential case by combining Eqs. (3.29) and (4.9). Both (4.24) and (4.25) have been verified numerically for the case of uniform $\phi(r)$.

Subdiffusive behavior is found in the mean square displacement of a particle starting from an initial configuration without long wavelength disorder.[12] This is given by (4.20) with $n = t' = 0$. For large t one obtains

$$\langle \zeta_0^2(t) \rangle = G_0(t, 0) \approx \sigma \sqrt{\frac{t}{\pi \mu_1}} = \frac{\mu_2}{\rho^2(\mu_1 - \mu_2)} \sqrt{\frac{\mu_1 t}{\pi}} \tag{4.26}$$

Using (3.43) and (4.6) this is seen to agree with the expression

$$\langle \zeta_0^2(t) \rangle = \sqrt{2/\pi}\,(\chi/\rho^2)\,\sqrt{D_{\mathrm{c}} t} \tag{4.27}$$

derived from hydrodynamic arguments.[3]

4.2.2. The Independent Jump Approximation. For the totally asymmetric simple exclusion process it is known[31, 2, 11] that the motion of a tagged particle in the stationary state follows a Poisson process, and therefore the tracer diffusion coefficient is simply equal to the mean speed $1-\rho$. Here we show that the expressions (4.23)–(4.25) for the ARAP are consistent with a similar independent jump picture.

Consider first the case of discrete time dynamics, where the random choice of the jump length δ_i is the only source of disorder, and therefore the tracer diffusion coefficient for independent jumps is equal to the variance of δ_i. For parallel update δ_i is a uniform random fraction of the particle headway u_i, hence $\langle \delta^2 \rangle - \langle \delta \rangle^2 = \mu_2 \langle u^2 \rangle - \mu_1^2/\rho^2$, which is easily checked to coincide with (4.24). For the backward sequential case δ_i is a random fraction of the intermediate state headway u_i'. Therefore, using Eqs. (3.28), (3.27) and (3.29),

$$\langle \delta^2 \rangle - \langle \delta \rangle^2 = \mu_2 \langle (u')^2 \rangle - \mu_1^2 \langle u' \rangle^2 = \frac{1}{(1-\mu_1)^2} \left(\frac{\mu_2 \langle u^2 \rangle}{1 - 2\mu_1 + \mu_2} - \frac{\mu_1^2}{\rho^2} \right) \tag{4.28}$$

which is also found to agree with (4.25).

In the continuous time case the random timing of jumps introduces an additional source of disorder. It is natural to assume, in analogy with the asymmetric exclusion process, that the jumps occur according to a Poisson process. In the independent jump approximation the particle displacement Δx in time t is then given by

$$\Delta x(t) = \sum_{l=1}^{n(t)} \delta^{(l)} \tag{4.29}$$

where $n(t)$ is a Poisson random variable with mean t and the jump lengths $\delta^{(l)}$ are independent random fractions of the (independent, random) particle headways. It is straightforward to show that the variance of Δx is

$$\langle (\Delta x)^2 \rangle - \langle \Delta x \rangle^2 = \langle \delta^2 \rangle \, t \tag{4.30}$$

thus in this case the independent jump approximation to D_{tr} is $\langle \delta^2 \rangle = \mu_2 \langle u^2 \rangle$ in agreement with (4.23).

5. SUMMARY AND OUTLOOK

We have presented results for two classes of particle systems on $\mathbb{R}$. The models considered in Section 2 have Poisson invariant measures and nonlinear current density relations (see Eqs. (2.8) and (2.14)). Time-dependent fluctuations in these models are therefore expected[4, 20] to be governed by the noisy Burgers (or Kardar–Parisi–Zhang[14]) equation, which is not amenable to simple analysis. By contrast, the asymmetric random average processes introduced in Section 3 have nontrivial invariant measures, but the linearity of the jump rules allows for a detailed study of dynamic properties (Section 4).

A central result for the ARAP is the dependence of the headway distribution (1.7) on the type of dynamics. The idea that parallel update reduces density fluctuations is familiar from earlier work on the asymmetric exclusion process and related models for traffic flow, however in that case the ordered sequential update produces the same (Bernoulli) invariant measure as the continuous time process.[27]

Our study suggests that the invariant measure of the continuous time ARAP displays an unusual combination of features: The two-point headway correlations factorize, the single particle headway distribution appears to be exactly given by the expression (1.7) derived under the assumption of pairwise independence, but nevertheless the product measure (1.3) is *not* invariant. Rajesh and Majumdar have found the same features in a larger class of models which interpolate between continuous time and parallel update.[26] It would be most interesting to find a simple "deformation" of the product measure which explains this behavior. The status of the product measure assumption for the ordered sequential update also remains to be clarified. The considerations of Section 3.2.2 indicate that it might be possible to exactly reduce this case to that of parallel update, for which the product measure is known to be invariant.[7]

Another interesting direction for future work is the introduction of quenched random inhomogeneities. In asymmetric exclusion models it is possible to find invariant product measures also in the presence of random jump rates associated with particles.[5, 21, 8, 9] For the ARAP with jump rates γ_i depending on the particle label i (the position i in the stick representation) preliminary numerical simulations indicate that the product measures discussed above do not persist. It is possible to write down a closed set of linear equations for the two-point function $\langle u_i u_j \rangle$ which depends on the disorder configuration $\{\gamma_i\}$ and which should yield insight into the emergence and nature of correlations. Here we merely remark that, since the mean speed of particle i is $\gamma_i \langle \delta_i \rangle$, stationarity implies $\langle u_i \rangle = C/\gamma_i$ where the constant C is fixed by the average headway. If the distribution of jump rates is chosen such that $\langle 1/\gamma_i \rangle$ exists, $C \to \langle 1/\gamma_i \rangle^{-1}$ in the limit of infinite system size, and all headways have a finite mean. Otherwise (e.g., for a uniform distribution of jump rates) arbitrarily large headways will open in front of the slowest particles, similar to the low density phase of asymmetric exclusion models with particlewise disorder.[21, 8, 9]

ACKNOWLEDGMENTS

We are much indebted to Bernard Derrida, Pablo Ferrari, Herve Guiol, Satya Majumdar, Gunter Schütz and Timo Seppäläinen for useful discussions and remarks. This work was supported by DAAD and CAPES within the PROBRAL programme. J.K. acknowledges the hospitality of IME/USP, São Paulo, and the Erwin Schrödinger Institute for Mathematical Physics, Vienna, where part of the paper was written. J.G. acknowledges the hospitality of Universität GH Essen during the early stages of the project.

REFERENCES

1. D. Aldous and P. Diaconis, Hammersley's interacting particle process and longest increasing subsequences, *Prob. Theory Relat. Fields* **103**:199–213 (1995).
2. R. Arratia, The motion of a tagged particle in the simple symmetric exclusion system on $\mathbb{Z}^1$, *Ann. Prob.* **11**:362–373 (1983).
3. H. van Beijeren, K. W. Kehr, and R. Kutner, Diffusion in concentrated lattice gases III. Tracer diffusion on a one-dimensional lattice, *Phys. Rev. B* **28**:5711–5723 (1983).
4. H. van Beijeren, R. Kutner, and H. Spohn, Excess noise for driven diffusive systems, *Phys. Rev. Lett.* **54**:2026–2029 (1985).
5. I. Benjamini, P. A. Ferrari, and C. Landim, Asymmetric conservative processes with random rates, *Stoch. Proc. Appl.* **61**:181–204 (1996).
6. P. Claudin, J.-P. Bouchaud, M. E. Cates, and J. P. Wittmer, Models of stress fluctuations in granular media, *Phys. Rev. E* **57**:4441–4457 (1998).
7. S. N. Coppersmith, C.-H. Liu, S. Majumdar, O. Narayan, and T. A. Witten, Model for force fluctuations in bead packs, *Phys. Rev. E* **53**:4673–4685 (1996).
8. M. R. Evans, Bose-Einstein condensation in disordered exclusion models and relation to traffic flow, *Europhys. Lett.* **36**:13–18 (1996).
9. M. R. Evans, Exact steady states of disordered hopping particle models with parallel and ordered sequential dynamics, *J. Phys. A* **30**:5669–5685 (1997).
10. S. Feng, I. Iscoe, and T. Seppäläinen, A class of stochastic evolutions that scale to the porous medium equation, *J. Stat. Phys.* **85**:513–517 (1996).
11. P. A. Ferrari and L. R. G. Fontes, Poissonian approximation for the tagged particle in asymmetric simple exclusion, *J. Appl. Prob.* **33**:411–419 (1996).
12. P. A. Ferrari and L. R. G. Fontes, Fluctuations of a surface submitted to a random average process, *El. J. Prob.* **3**:1–35 (1998).
13. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, Amsterdam, 1992).
14. M. Kardar, G. Parisi, and Y. C. Zhang, Dynamic scaling of growing surfaces, *Phys. Rev. Lett.* **56**:889–892 (1986).
15. C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, Berlin/Heidelberg, 1999).
16. S. Krauss, P. Wagner, and C. Gawron, Continuous limit of the Nagel–Schreckenberg model, *Phys. Rev. E* **54**:3707–3712 (1996).
17. S. Krauss, P. Wagner, and C. Gawron, Metastable states in a microscopic model for traffic flow, *Phys. Rev. E* **55**:5597–5602 (1997).
18. S. Krauss, Microscopic traffic simulation: Robustness of a simple approach, in *Traffic and Granular Flow '97*, M. Schreckenberg and D. E. Wolf, eds. (Springer, Singapore, 1998), pp. 269–283.
19. J. Krug, On the shape of wedding cakes, *J. Stat. Phys.* **87**:505–518 (1997).
20. J. Krug and H. Spohn, Kinetic roughening of growing surfaces, in *Solids Far From Equilibrium*, C. Godreche, ed. (Cambridge University Press, Cambridge, 1991), pp. 479–582.
21. J. Krug and P. A. Ferrari, Phase transitions in driven diffusive systems with random rates, *J. Phys. A* **29**:L465–L471 (1996).
22. T. M. Liggett, *Interacting Particle Systems* (Springer, Berlin, 1985).
23. T. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* (Springer, Berlin/Heidelberg, 1999).
24. C.-H. Liu, S. R. Nagel, D. A. Schecter, S. N. Coppersmith, S. Majumdar, O. Narayan, and T.A. Witten, Force fluctuations in bead packs, *Science* **269**:513–515 (1995).
25. S. N. Majumdar, S. Krishnamurthy, and M. Barma, Nonequilibrium phase transition in a model of diffusion, aggregation and fragmentation, *J. Stat. Phys.* **99**:1 (2000).

26. R. Rajesh and S. N. Majumdar, Conserved mass models and particle systems in one dimension, Preprint, cond-mat/9910206.
27. N. Rajewsky, L. Santen, A. Schadschneider, and M. Schreckenberg, The asymmetric exclusion process: Comparison of update procedures, *J. Stat. Phys.* **92**:151–194 (1998).
28. P. Rosenau, Fast and superfast diffusion processes, *Phys. Rev. Lett.* **74**:1056–1059 (1995).
29. G. Schütz, Exact tracer diffusion coefficient in the asymmetric random average process, Preprint (1999).
30. T. Seppäläinen, A microscopic model for the Burgers equation and longest increasing subsequences, *El. J. Prob.* **1**:1–51 (1996).
31. F. Spitzer, Interaction of Markov processes, *Adv. Math.* **5**:246–290 (1970).
32. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin/Heidelberg, 1991).